

SOME MINIMAL ELEMENTS FOR A PARTIAL ORDER OF PRIME KNOTS

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ABSTRACT. A partial order on the set of prime knots can be defined by the existence of an epimorphism between knot groups. We prove that all the prime knots with up to 6 crossings are minimal. We also show that each fibered knot with the irreducible Alexander polynomial is minimal.

1. INTRODUCTION

Let K be a non-trivial prime knot in S^3 and $G(K)$ the knot group of K , which is the fundamental group of $S^3 \setminus K$. For two knots K, K' , we write $K \geq K'$ if there exists an epimorphism $\varphi : G(K) \rightarrow G(K')$ which preserves meridians. This relation \geq gives a partial order on the set of prime knots. In [6, 7, 9, 10], this partial order is determined for the set of the prime knots with up to 11 crossings. It showed that the knots $3_1, 4_1, 5_1, 5_2, 6_1, 6_2, 6_3$ are minimal in this set. In this paper, we prove that these knots are minimal in the set of all prime knots. Here we call a knot K to be minimal if $G(K)$ does not surject onto the knot group of any non-trivial knot. Note that a knot group always admits an epimorphism onto the knot group of the trivial knot.

Theorem 1.1. *The knots $3_1, 4_1, 5_1, 5_2, 6_1, 6_2, 6_3$ are minimal in the set of all prime knots.*

The same statement holds for another partial order derived from an epimorphism between knot groups (not necessary to preserve meridians).

2. PROOF OF RESULTS

First, we review the following fact on a fibered knot.

Proposition 2.1 ([8]). *If $K \geq K'$ and K is fibered, then K' is also fibered and $g(K)$ is greater than or equal to $g(K')$, where $g(K)$ is the genus of K .*

Proof. An epimorphism between knot groups induces an epimorphism between their commutator groups. Since K is fibered, the commutator subgroup $[G(K), G(K)]$ is the free group of rank $g(K)$. This implies $[G(K'), G(K')]$ is also finitely generated. By using the fibration theorem by Stallings [11], we see that $[G(K'), G(K')]$ is isomorphic to the fundamental group of a compact surface S and that K' is a fibered knot with the fiber S . Further we have $g(K) \geq g(K')$. \square

Corollary 2.2 ([8]). *The knots 3_1 and 4_1 are minimal.*

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Proof. We assume that $3_1 \geq K$ and $K \neq 3_1$. By Proposition 2.1, K is a fibered knot of genus one or the trivial knot. It is well known that the fibered knots of genus one are only 3_1 and 4_1 . If there exists an epimorphism $G(K) \rightarrow G(K')$, then the Alexander polynomial of K can be divided by that of K' (see [5]). However, the Alexander polynomial of 3_1 cannot be divided by that of 4_1 . Hence it follows that $3_1 \not\geq 4_1$. Similarly, it is shown that 4_1 is minimal. \square

Furthermore, we obtain the following.

Proposition 2.3. *Let K, K' be fibered knots of the same genus. If $K \geq K'$, then $K = K'$.*

Proof. By the proof of Proposition 2.1, there exists an epimorphism between the commutator subgroups of the knot groups $G(K), G(K')$. By assumption, the commutator subgroups $[G(K), G(K)], [G(K'), G(K')]$ are the free groups of the same rank. This implies the epimorphism $[G(K), G(K)] \rightarrow [G(K'), G(K')]$ is an isomorphism, since the free group is Hopfian. Then the epimorphism $G(K) \rightarrow G(K')$ is also an isomorphism. Therefore we obtain that $K = K'$ for prime knots K, K' . \square

Corollary 2.4. *The knots $5_1, 6_2$, and 6_3 are minimal.*

Proof. It is known that 5_1 is a fibered knot of genus two. Suppose $5_1 \geq K$, then K is a fibered knot of genus one or two. If the genus of K is one, then K is 3_1 or 4_1 . However, the Alexander polynomial of 5_1 can be divided by neither that of 3_1 nor that of 4_1 . Then the genus of K is two. It implies $K = 5_1$ by Proposition 2.3. By the similar argument, 6_2 and 6_3 are also minimal. \square

Boileau-Boyer-Reid-Wang [2] and Boileau-Boyer [1] studied epimorphisms of 2-bridge knot groups. As an application, we get some minimal elements.

Proposition 2.5. *The knots 5_2 and 6_1 are minimal.*

Proof. By the result of Boileau-Boyer-Reid-Wang [2], if a 2-bridge knot group surjects onto $G(K)$, then K is a 2-bridge knot or the trivial knot. Moreover, Boileau-Boyer [1] showed that if there exists an epimorphism between 2-bridge knot groups, namely $S(p, q) \geq S(p', q')$, then $p = kp'$ where $k > 1$. Then $5_2 = S(7, 3)$ is minimal. Besides, it is easy to see that the Alexander polynomial of $6_1 = S(9, 7)$ cannot be divided by that of $3_1 = S(3, 1)$. Therefore there exists no epimorphism from $G(6_1)$ onto $G(3_1)$. \square

3. FIBERED KNOT WITH IRREDUCIBLE ALEXANDER POLYNOMIAL

Proposition 3.1. *If K is a fibered knot and the Alexander polynomial $\Delta_K(t)$ is irreducible over \mathbb{Z} , then K is minimal.*

Proof. Let K be a fibered knot whose Alexander polynomial is irreducible over \mathbb{Z} . Suppose that $K \geq K'$ and that K' is non-trivial. Then $\Delta_K(t)$ can be divided by $\Delta_{K'}(t)$ and K' is fibered by Proposition 2.1. Here the Alexander polynomial of K' is not trivial. Since $\Delta_K(t)$ is irreducible, we have $\Delta_K(t) = \Delta_{K'}(t)$. It implies that the genus of K is the same as that of K' . Therefore $K = K'$ by Proposition 2.3. \square

Example 3.2. We consider 8_{16} , which is a fibered knot of genus 3. Since 8_{16} is a 3-bridge knot, we cannot apply the similar argument to Proposition 2.5. The

Alexander polynomial is $t^6 - 4t^5 + 8t^4 - 9t^3 + 8t^2 - 4t + 1$, which is irreducible over \mathbb{Z} . Then we obtain that 8_{16} is minimal. Similarly, we can show that 8_{17} is also minimal.

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